Instant Insanity: Uniqueness and Existence of Solutions

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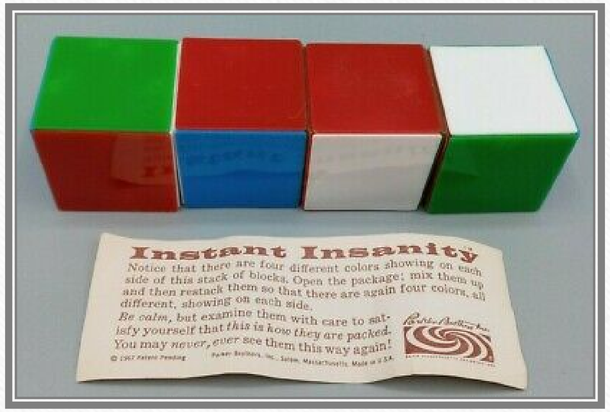
What is Instant Insanity?

- Instant Insanity is a stacking cube puzzle
- It's on Ebay!
- Each cube has up to 4 colors
- Each side of the cube tower must have one of each color face



1967 Parker Brothers Instant Insanity

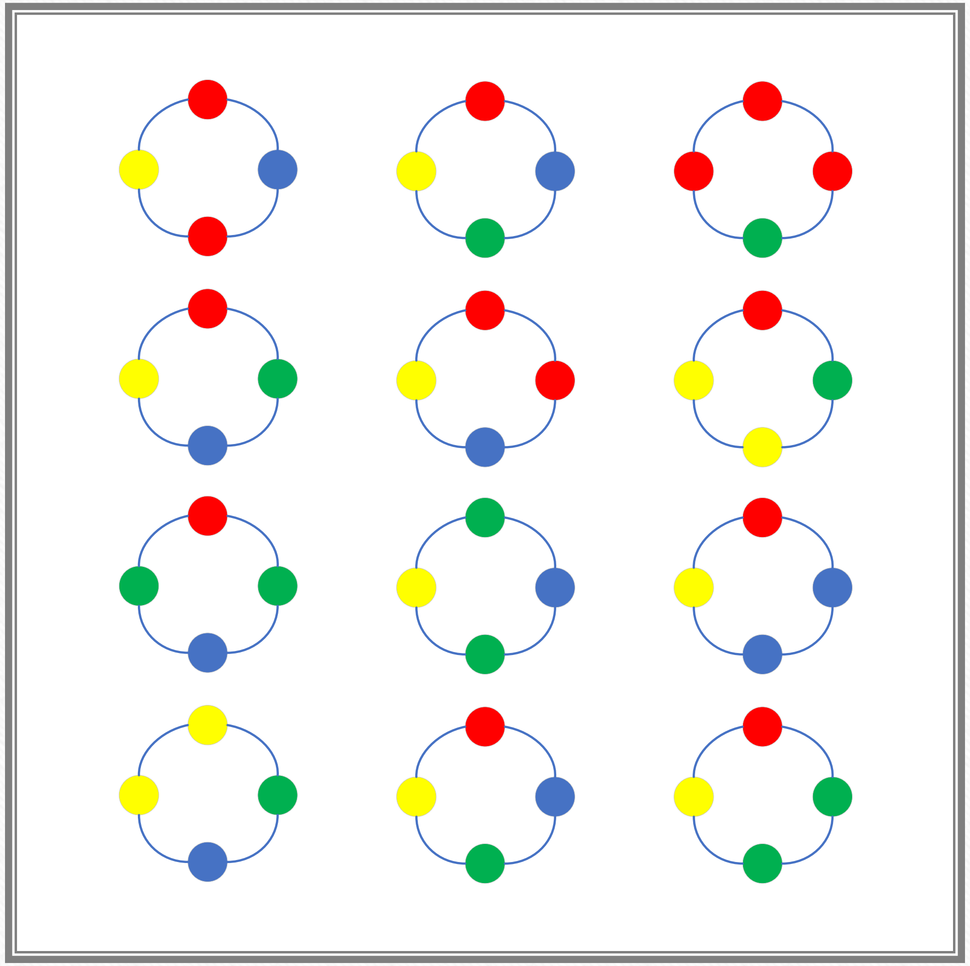
Example of Solved Puzzle

Four Research Goals

1. How can I solve a specific puzzle if 4 cubes are given to me?
2. Can I prove that my solution is the only solution to the puzzle?
3. Existence: In general, what leads a tower of cubes to have a solution?
4. Uniqueness: In general, what leads a tower of cubes to have a unique solution?

Excluding Faces

- Each cube has three pairs of opposing faces
- One pair can be "excluded" from each cube
  - Can't "see" a pair on the top/bottom of the cube
- Goal: exclude pairs so that tower has 4 Yellow, 4 Red, 4 Blue, 4 Green faces
- Only tells us we have the right number of colors for a solution
- We still need a way to visualize sides & how they touch!

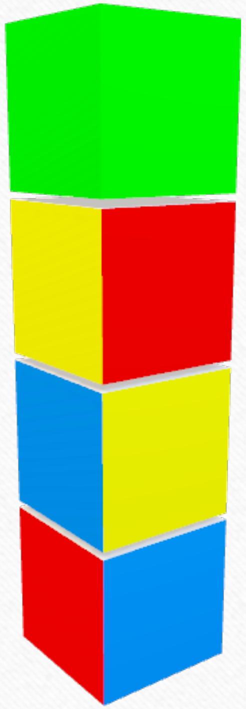


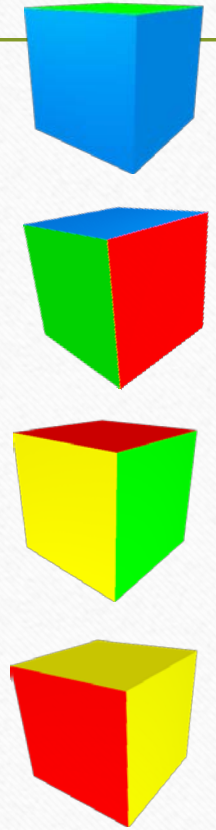
Necklaces

- Represent the sides of a cube
- A solved tower is 4 stacked necklaces
- A mathematical object used by combinatorialists to solve problems
- Can rotate or flip necklace

Equivalence Classes

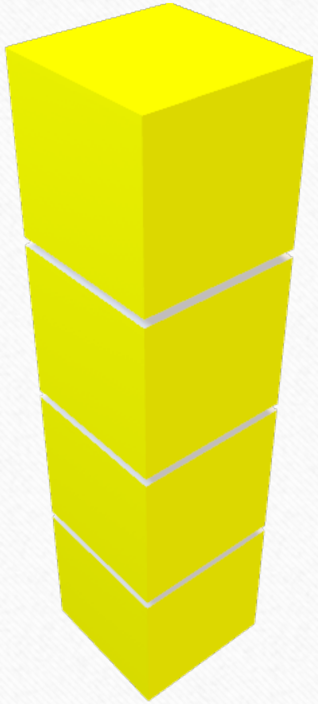
- “Dumb solutions”
- Spin class: 4
- Flip class: 2
- “Swap” class:
  4!= 1*2*3*4=24
- 192 total
- Anything not in these classes is considered a separate solution!





Existence: Generalizations

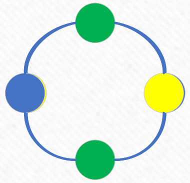
- The “No Solution Tower”
- There can be no more than 12 of the same color
  - If there are twelve, 4 must be on the sides
  - 8 must be “excluded” in same color pairs
- There can be no more than 8 of the same “free” color
  - Free: not in an opposing pair with the same color

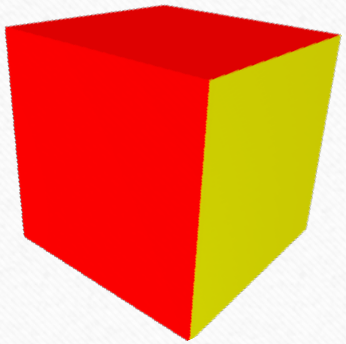


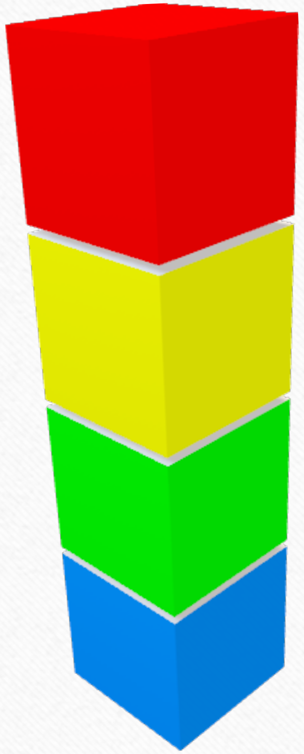
Uniqueness: Generalizations

- The "All Solution Tower"
- There must be only one way to form the "solving" necklace of a cube
- **Most important result:** If it's a unique solution, no necklace can have an opposing pair of the same color! It makes a second solution not created through equivalence class movements*

*A proof is provided!







Source: A Gwen Fisher Original Illustration
https://www.etsy.com/shop/gwenbeads

Uniqueness: Generalizations

- No cube can have 5 or more faces of the same color
- Due to "The Pigeonhole Principle"
  - If there are 5 of the same color, it will make a necklace that has an opposing pair of the same color

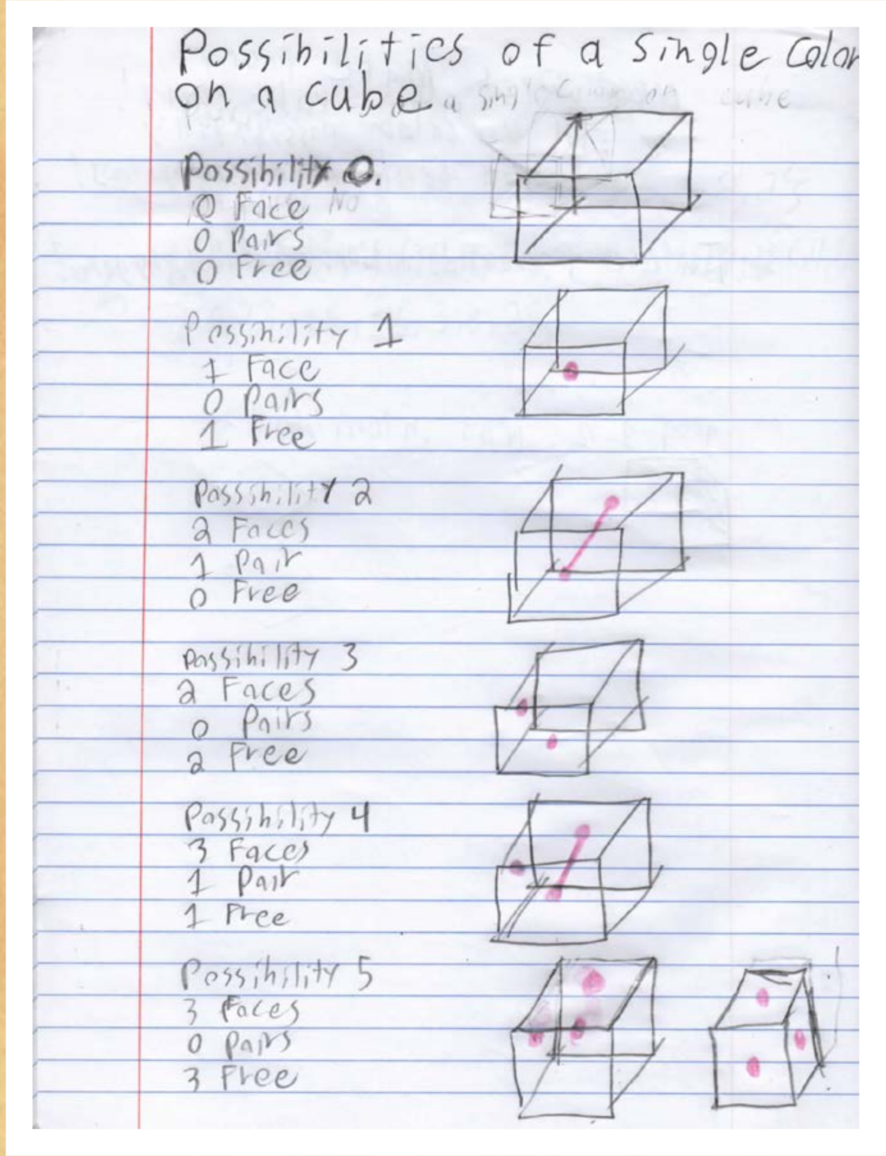
Single Color Possibilities on a Single Cube

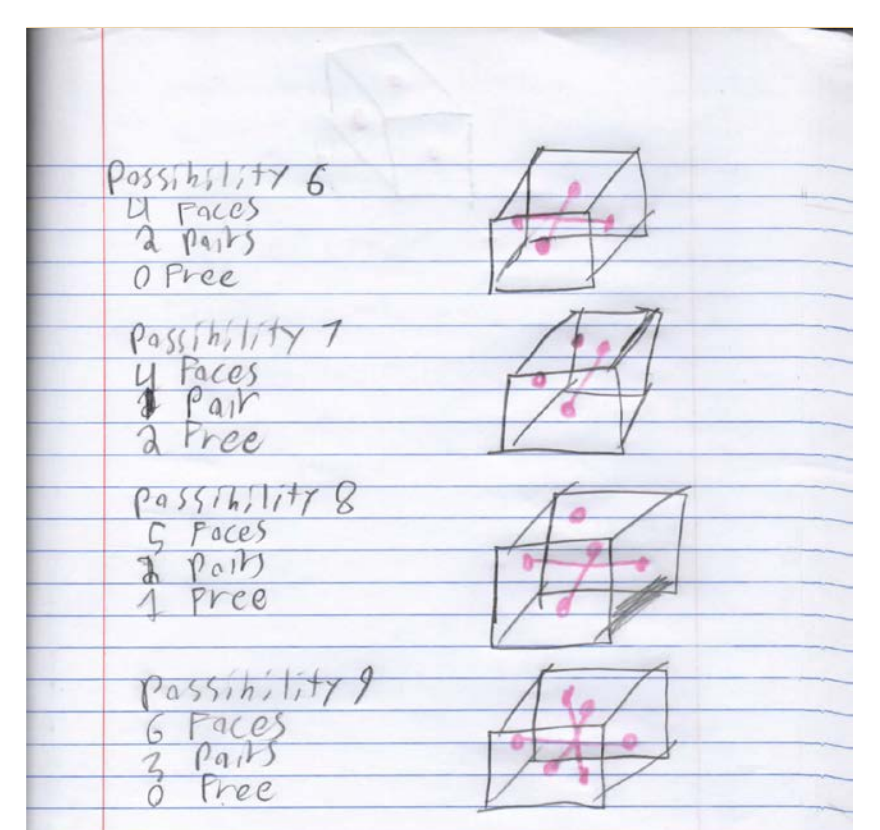
Questions to answer:

- How many different layouts of colors are there?
- How many layouts will give solutions?
- How many of those solutions will be unique?

Strategy:

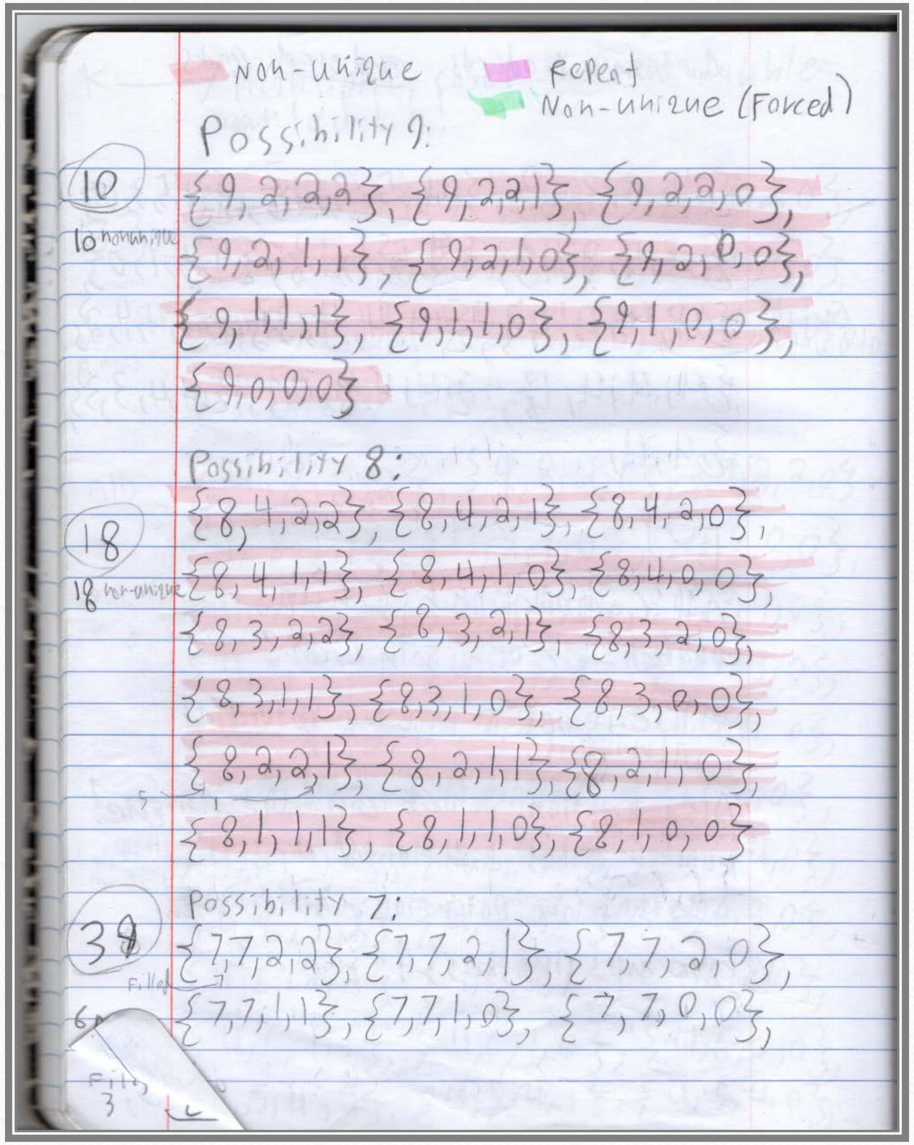
- Start by just looking at a single color on a single cube!





Possibility 6, 8, and 9 must be non-unique! They have same color pairs that cannot be excluded.

Iterating Possibilities Through the Tower



Iterating Possibilities Through the Tower

- Fun fact: the bracketed groups of possibilities are called “multisets”
- 2 ways to have a non-unique solution
  - Inherently non-unique: A single cube must have a same-color pair on the sides
    Ex: {9,0,0,0}
  - Forced non-unique: The combination of cubes “force” a same color pair to the sides
    Ex: {0, 0, 2, 2}

Probability of Existence and Uniqueness

- 715 combinations
  - I found a pattern while counting, then used a recursive formula in Sigma notation to find the rest! Ask me about it.
- 183 are potentially valid (25.6%)
  - "Potentially": Depending on combination with the other 3 colors
- 94 are potentially valid and non-unique (13.2%)
- 89 are potentially valid and potentially unique (12.4%)

Further Research Considerations

- Is there a way to solve the cube using matrices?
- Is there a way to show how more than one color of the cube interact and iterate through the tower?
- A necklace stacking game? Hexagon necklaces, “n-gon” necklaces

Questions?

Backup Slides

Recurrence Relation Conjecture

- We have 4 “slots” in a multiset, with n being our number of possibilities (single color possibilities on a single cube)
- To find all the non-repeating combinations for 4 consecutive slots,

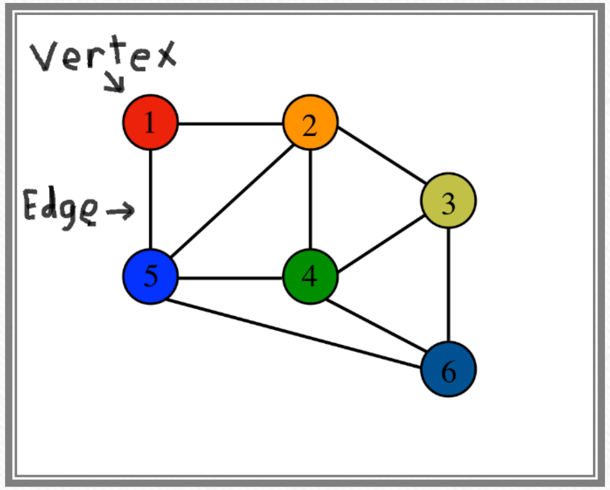
$$C(n) = \left(\sum_{i=1}^{n}(n-i)(i+1)\right) + C(n-1),$$

where C(n) is the number of total non-repeating combinations for a given n, and

$$C(0) = 0$$

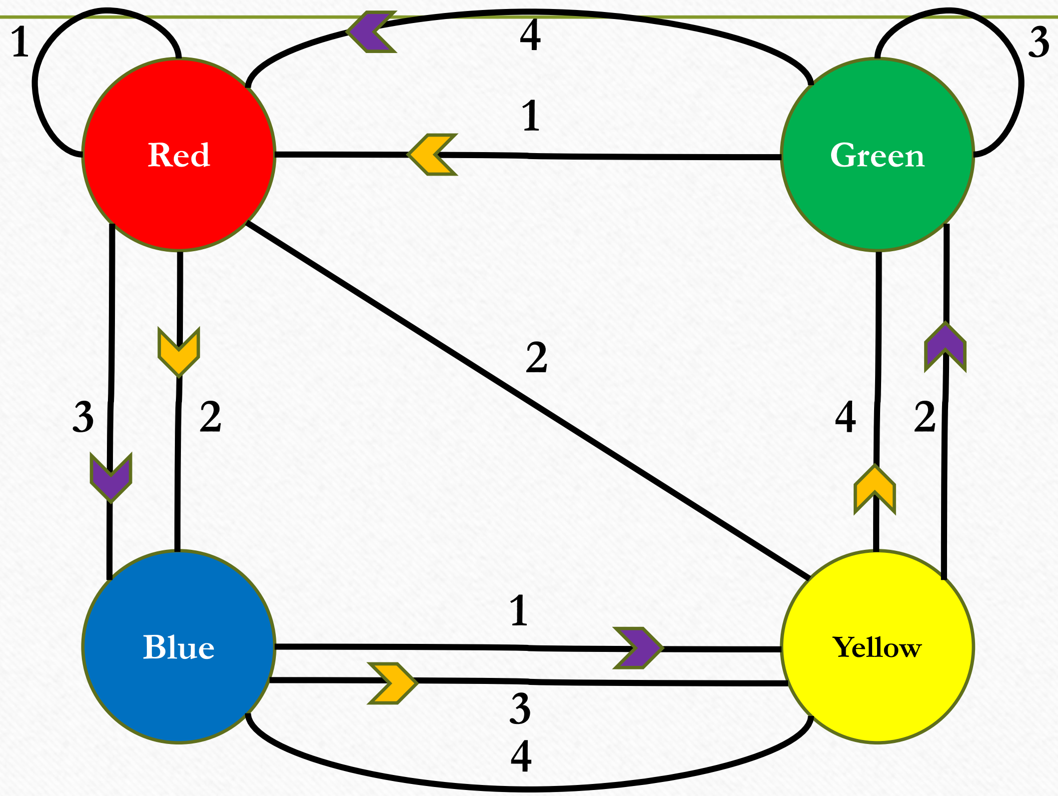
Graphs

- A graph is a mathematical object made of edges and vertices
- Shows which objects are connected to each other, and how many times
- Why: to easily make a "map" of the problem!
  - The tower will be encoded as a graph
- Directed graph: Shows which direction an object is connected to another



Directed Graph Animation

I found 4 valid ways to exclude cube faces



Rules:

1. Each cube must have North/South and East/West Arrow
2. Each color must have one of each arrow pointed at it

This was the only solution